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FLOW OF A THIN LAYER OF A VISCOUS LIQUID OVER A DRY SURFACE

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The problem of flow of a thin layer of a viscous liquid over a dry surface is formulated (ignoring surface tension). Examples of calculations are given. The results are compared with a solution using an appropriate one-dimensional model that admits an exact solution. Singularities of the solution are analyzed.

1. Equations. The equations of a thin layer of a viscous liquid (a long-wave approximation) ignoring surface tension have the form [1] (Fig. 1)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial z} + g\frac{\partial(h\cos\theta)}{\partial x} = \nu\frac{\partial^2 u}{\partial z^2} + g\sin\theta; \tag{1.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} = 0, \tag{1.2}$$

where $u = u(t,x,z)$, $v = v(t,x,z)$, $h = h(t,x)$, $0 \leqslant z \leqslant h$, $0 \leqslant x \leqslant l(t)$, and $x \sim l(t) \Rightarrow h \sim 0$.

Equations (1.1) and (1.2) are supplemented by the following boundary conditions [1]:
on the rigid surface ($z = 0$),

$$u = v = 0; \tag{1.3}$$

and on the free surface ($z = h$),

$$\frac{\partial u}{\partial z} = 0, \quad \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = v. \tag{1.4}$$

Model (1.1)–(1.4) will be referred to as a two-dimensional model.

The conditions and singularities at the point of contact of the free and solid surfaces are discussed below. For $x = 0$, we set $u = \varphi(t,z)$, and for $t = 0$ we have $u = u_0(x,z)$ and $h = h_0(x)$. It is assumed that $u > 0$.

In the formulation of the initial and boundary conditions, the distribution of the velocity u in z obeys a parabolic law.

2. One-Dimensional Model. Before discussing the singularities of model (1.1)–(1.4), it is useful to consider the one-dimensional model. Assuming that for $z \geqslant 0$ the velocity distribution corresponds to a parabolic law and integrating (1.1) and (1.2) for z going from $z = 0$ to $z = h$, we obtain the equations [2–4]

$$\frac{\partial h}{\partial t} + \frac{\partial(hU)}{\partial x} = 0; \tag{2.1}$$

$$\frac{\partial(hU)}{\partial t} + \frac{\partial}{\partial x}\left(\frac{6}{5}hUU + \frac{gh^2}{2}\cos\theta\right) = gh\sin\theta - \frac{3\nu U}{h} \qquad \left(U = \frac{q}{h}, \quad q = \int\limits_0^h u\,dz\right). \tag{2.2}$$

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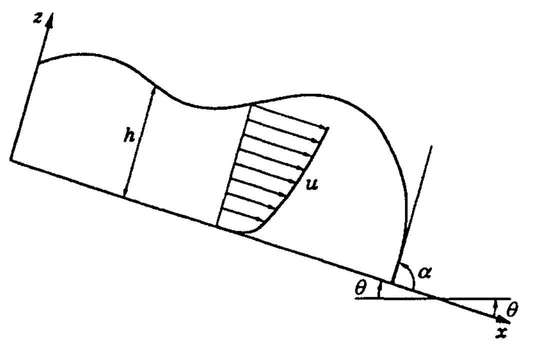

Fig. 1

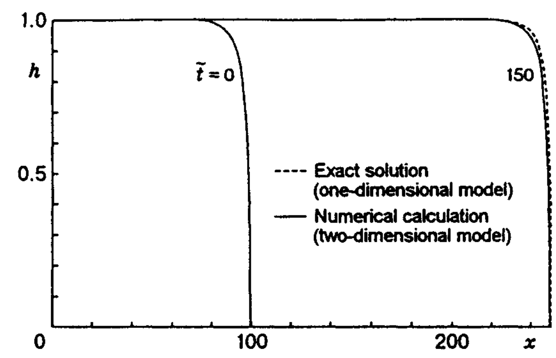

Fig. 2

Noteworthy is the presence of a singularity in the term that describes friction: for $h \sim 0$ and $U \neq 0$, the friction becomes indefinitely large. This singularity has a determining effect on the character of the free surface near the point of contact with the solid surface.

System (2.1) and (2.2) admits a simple and useful exact solution (previously, a similar solution was obtained in hydraulics [4]). Let

$$U = \text{const} > 0 = gH^2 \sin\theta/(3\nu), \qquad \theta > 0.$$

We introduce the following dimensionless quantities:

$$\tilde{x} \equiv \frac{x}{h}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{h} \equiv \frac{h}{H}, \quad \tilde{U} \equiv \frac{U}{V}, \quad T \equiv \frac{H}{V}, \quad V \equiv gH^2\frac{\sin\theta}{3\nu}, \quad \text{i.e., } \tilde{U} = 1.$$

Setting $\xi \equiv \tilde{x} - \tilde{t}$, we have $h = h(\xi)$. In the given case, it is also convenient to set

$$\mathrm{Re} \equiv \frac{UH}{\nu} = \frac{gH^3}{3\nu^2}\sin\theta. \tag{2.3}$$

As a result, we obtain

$$-A\tilde{h} + \frac{A-1}{2}\ln(1+\tilde{h}) - \frac{1+A}{2}\ln(1-\tilde{h}) = -\frac{15}{\mathrm{Re}}(\xi - \xi_0) = -\frac{15}{\mathrm{Re}}(\tilde{x} - \tilde{x}_0) - \tilde{t}_0, \quad A \equiv \frac{15}{\imath\mathrm{Re}}. \tag{2.4}$$

3. Singularities and Conditions at the Point of Contact. Under the assumption that $U(t,x) > 0$ (the function U is smooth) for $x < l(t)$, the natural boundary condition at the point of contact $x = l(t)$ is written as

$$x \to l \Rightarrow h \to 0, \qquad \frac{dl}{dt} = \lim_{x \to l} U. \tag{3.1}$$

From (2.4) it follows that for $\tilde{h} \sim 0$, we have

$$\frac{d\tilde{h}}{d\tilde{x}} \sim \frac{1}{\sqrt{|\tilde{l} - \tilde{x}|}}, \qquad \tilde{h} \sim \sqrt{|\tilde{l} - \tilde{x}|}. \tag{3.2}$$

For (2.1) and (2.2), the singularity (3.2) has a general character. Indeed, we assume that for $x < l$, the function U is smooth and $\lim_{x \to l} U = \hat{U} > \alpha$ ($\alpha = \text{const} > 0$). According to (3.1), we have $dl/dt = \hat{U} > 0$. Taking into account (2.1) and retaining the basic terms for $h \sim 0$ in (2.2), we have

$$\frac{\partial h}{\partial t} + \hat{U}\frac{\partial h}{\partial x} = 0, \qquad \frac{1}{5}\hat{U}\hat{U}h\frac{\partial h}{\partial T} \sim -3\nu\hat{U},$$

whence

$$h \sim \sqrt{\frac{30\,\nu|l - x|}{\hat{U}}}, \quad \frac{\partial h}{\partial t} \sim \sqrt{\frac{15\nu}{\hat{U}}\frac{1}{|l - x|}}. \tag{3.3}$$

For the two-dimensional model (1.1)–(1.4), using a simple line of reasoning we also obtain a singularity of the type of (3.2). In this case, it is significant that in a neighborhood of the point of contact, $U(x) > \alpha$ ($\alpha = \text{const} > 0$). Since the reasoning that leads to (3.2) is not too rigorous, it is expedient to omit it and regard (3.2) as a probable hypothesis confirmed by a computational experiment (Fig. 2 and Example 1).

Apparently, in the absence of surface tension, the singularity (3.2) also holds for the two-dimensional Navier–Stokes equations provided that in a neighborhood of the point of contact with the solid surface $x = l(t)$, the condition

$$U(x) = \frac{1}{h}\int_0^h u\,dz > 0 \qquad (x < l) \tag{3.4}$$

is satisfied (the boundary moves to the right). In all these cases with no surface tension and with satisfaction of condition (3.4) in a neighborhood of the point of contact, the natural boundary condition is similar to (3.1), i.e., has the form

$$x \to l \Rightarrow h \to 0, \qquad \frac{dl}{dt} = \lim_{x\to l} U. \tag{3.5}$$

If the liquid flows on the average from the point of contact, i.e., if in a neighborhood of the point $x = l(t)$, the condition opposite to (3.4),

$$U(x) = \frac{1}{h}\int_0^h u dz \leqslant 0 \qquad (x < l), \tag{3.6}$$

is satisfied, it is most likely that the point of contact is completely attached to the solid surface, i.e., (3.5) should be replaced by the condition

$$x \to l(t) \Rightarrow h \to 0, \qquad \frac{dl}{dt} = 0. \tag{3.7}$$

Thus, in the above cases with no surface tension, the boundary condition is of the same type: if in a neighborhood of the point of contact, (3.4) holds, condition (3.5) is specified, and if (3.6) holds, condition (3.7) is specified.

In addition, if $\lim_{x\to l} U > 0$ and the function $U(x,t)$ is smooth in a neighborhood of the point of contact, there is a singularity of the type (3.2). Thus, the angle α between the free surface and the solid surface is equal to $\pi/2$.

For numerical solution of the problem, conditions (3.4)–(3.7) are not convenient. Therefore, in the calculations it was assumed that $h = \varepsilon > 0$ at $t = 0$ and $x \geqslant l(0)$, i.e., a layer of a rather shallow depth ($\varepsilon > 0$) was "poured." After that, a through-calculation procedure was used (for the two-dimensional model, it was combined with a splitting method). But this also involves certain computational difficulties because in the main flow region for $x < l$ and in the region of "poured" liquid for $x > l$, the characteristic sizes of the difference grid are markedly different; the difficulties were overcome using the implicit difference schemes described in [5].

This computational method of a "poured" shallow layer appears to be physically correct in the absence of surface tension. However, in the presence of the latter, the correctness of the method is questionable.

Remark. It follows from the aforesaid that the introduction of surface tension radically changes the situation. With accurate allowance for surface tension and the attachment condition (1.3), the two-dimensional model of a thin layer (1.1), (1.2) appears to be physically and mathematically incorrect in the presence of contact between the free and solid surfaces.

For two-dimensional Navier–Stokes equations, it was proved [6] that for $0 < \alpha < \pi$ (see Fig. 1), the free-surface conditions and the attachment condition are incompatible. Nevertheless, physically the situations $\alpha = 0$ and $\alpha = \pi$, appear to be possible for complete Navier–Stokes equations. If $\alpha = 0$, then for $x \sim l$, the liquid rolls on the solid surface as a wheel [in particular, this means that there is a singularity for the curl of ω as $z \to 0$ and $x \to l(t)$]. For $\alpha = \pi$, the point of contact is most likely to be attached to the solid wall, so that $dl/dt = 0$ and $\lim_{x\to l} u = 0$ in this case.

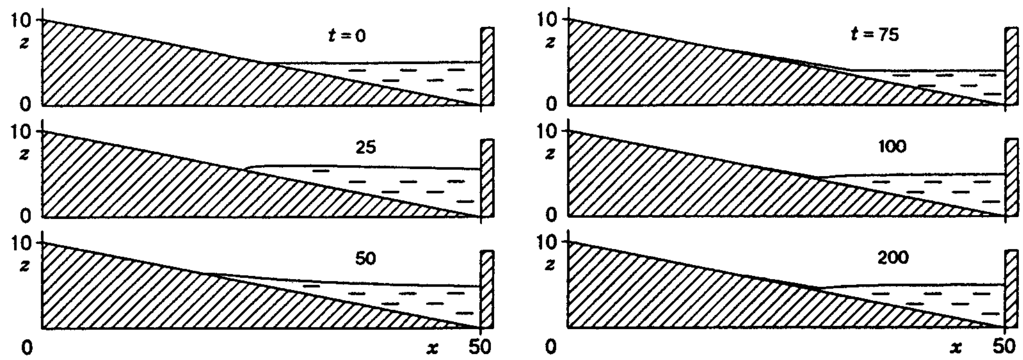

Fig. 3

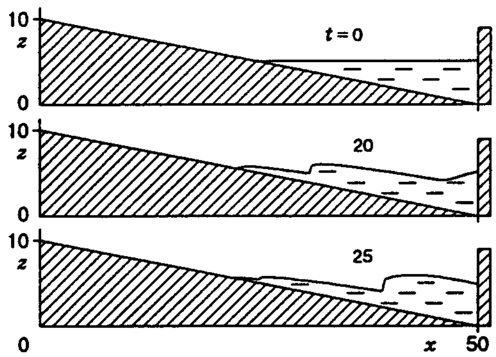

Fig. 4

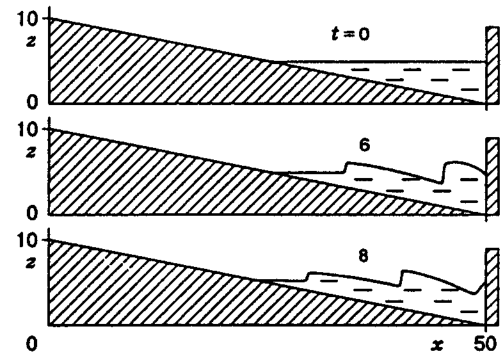

Fig. 5

Because of the difficulties related to the compatibility of attachment conditions and free-surface conditions if surface tension is taken into account, attachment conditions are replaced by slip conditions in some papers, and this regularizes the problem mathematically (see, e.g., [5, 6]). However, the physical correctness of this method is questionable.

4. Examples of Calculations. We give examples of calculations for the two-dimensional model (1.1)–(1.4).

Example 1. *Comparison of the Two-Dimensional and One-Dimensional Models.* For the one-dimensional model, we constructed the exact solution (2.4), which has points of contact between the free and solid surfaces with a tangency angle $\alpha = \pi/2$. It is of interest to compare this solution with a calculation using the two-dimensional model. For $\tilde{t} = 0$ and $\tilde{x} \leqslant 100$ (in dimensionless variables), the initial profile $h(x)$ was taken in accordance with (2.4), i.e., $\tilde{h}_0 = \varepsilon = 0.001$ for $\tilde{x} > 100$, and $\mathrm{Re} = 10$ [see (2.3)]. The solutions for $\tilde{t} = 150$ are compared in Fig. 2. Good agreement is observed everywhere except for the wave front, although the travel time is practically the same in both models.

The difference at the front suggests some inaccuracy of the one-dimensional model (2.1) and (2.2), probably because the assumption of parabolicity of the velocity distribution at the wave front is not quite valid.

Example 2. *Incidence of a Wave on a Shore.* Figures 3–5 show examples of calculation of the incidence of a wave on a shore using the two-dimensional model. For $x = 50$, the following law of variation of the free-surface mark was specified: $\tilde{z} = \tilde{z}_0 + A \sin 2\pi\tilde{t}/T_0$. For a large period ($T_0 = 100$), the wave runs smoothly on the shore and runs off smoothly (Fig. 3). With decrease in the period T_0, a sequence of intermittent waves (breakers) forms. Figure 4 gives an example of the calculation for $T_0 = 10$. The calculations show that, with a further decrease in T_0 and/or increase in the amplitude A, the amount of breakers increases (Fig. 5).

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